

Homework 11

Math 121A, Fall 2025.

Due: Wednesday, Nov 12, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. We should get comfortable with the spaces $\ell^2(\mathbb{Z})$ and $L^2([-\pi, \pi])$ we defined in lecture, which were both defined by their norms. Give examples of...

- (a) ... a function $f : [-\pi, \pi] \rightarrow \mathbb{C}$ which is in $L^2([-\pi, \pi])$.
- (b) ... a function $f : [-\pi, \pi] \rightarrow \mathbb{C}$ which is not in $L^2([-\pi, \pi])$.
- (c) ... a function $f : [-\pi, \pi] \rightarrow \mathbb{C}$ which diverges at 0, but is still in $L^2([-\pi, \pi])$.
- (d) ... a sequence $u : \mathbb{Z} \rightarrow \mathbb{C}$ which is in $\ell^2(\mathbb{Z})$.
- (e) ... a sequence $u : \mathbb{Z} \rightarrow \mathbb{C}$ which is not in $\ell^2(\mathbb{Z})$.

Problem 2. (*Simultaneous diagonalization and Heisenberg uncertainty, $\dim \mathcal{H} = n$*)

Consider the following pair of matrices, expressed in the standard basis of $\mathbb{C}^2$:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- (a) Let A and B be two diagonalizable linear operators on $\mathbb{C}^n$ and let A have all eigenvalues be distinct with multiplicity 1. Show that $AB = BA$ if and only if A, B are simultaneously diagonalizable, meaning they are both diagonal in a basis of shared eigenvectors.
Bonus: this equivalence is still true even if A does not have distinct eigenvalues. Prove that $AB = BA$ if and only if A, B are simultaneously diagonalizable.
- (b) We can measure failure to commute using the commutator $[A, B] := AB - BA$. Show that $[X, Z] \neq 0$. What does this tell us about simultaneous diagonalization?
- (c) By the spectral theorem, since $X = X^\dagger$, X is unitarily diagonalizable. Diagonalize X and verify that the change of basis matrix is unitary.
- (d) Compute the matrix for the Z in the eigenbasis of X . Is it diagonal?

Problem 3. (*Simultaneous diagonalization and Heisenberg uncertainty, $\dim \mathcal{H} = \infty$*)

- (a) Let $X : L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$ be the “position” operator and $P : L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$ the “momentum” operator,¹ defined by

$$X(f) = xf(x), \quad P(f) = -i \frac{d}{dx} f(x).$$

¹Just pretend we can take a derivative of any function in $L^2([-\pi, \pi])$. Technically, momentum is a densely defined self adjoint operator.

Clearly X is a multiplication (“diagonal”) operator. Show that the commutator $[X, P] \neq 0$, and so these operators may not be simultaneously diagonalized.

- (b) Parseval’s theorem states that the map $\hat{\cdot} : L^2([-\pi, \pi]) \rightarrow \ell^2(\mathbb{Z})$ which sends $f \mapsto \hat{f}$ is a unitary map. Unitary maps send orthonormal bases to orthonormal bases. Take the orthonormal basis $\{f_m(x) = e^{imx} : m \in \mathbb{Z}\}$ and compute the set of vectors $\{\hat{f}_m : m \in \mathbb{Z}\}$. Is this an orthonormal basis of $\ell^2(\mathbb{Z})$?
- (c) Show that the change of basis $\hat{\cdot}$ “diagonalizes” the momentum operator $P : L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$, $P = -i\frac{d}{dx}$, i.e. unitary conjugation turns it into a multiplication operator on $\ell^2(\mathbb{Z})$.

Problem 4. (*The Discrete Fourier Transform*)

A central theme from Fourier analysis is that the Fourier transform diagonalizes translations (and thus derivatives). Here we walk through a discrete example.

Consider a ring with n locations $0, 1, \dots, n-1$, so the $(n-1)^{th}$ position is next to the 0^{th} position (this is called “periodic boundary conditions”). Let $v = (v_0 \ v_1 \ \dots \ v_{n-1})^T$ be a vector with complex entries corresponding to these locations. The *translation operator* $S : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the linear map given by

$$(Sv)_x = v_{x-1}, \quad x = 0, 1, \dots, n-1,$$

where you should treat x as having periodic boundary conditions.

The *discrete Fourier transform* is the linear map $U : \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by

$$(Uv)_k = \frac{1}{\sqrt{n}} \sum_{x=0}^{n-1} v_x e^{-i2\pi xk/n}$$

- (a) Compute the matrices $[S]$ and $[U]$ in the standard basis. (*It may be useful to introduce the number $\omega := e^{-i2\pi/n}$.*)
- (b) Verify that U is unitary with respect to the dot product $\langle v, w \rangle = \sum_{x=0}^{n-1} \bar{v}_x w_x$ in any way you prefer. (*Hint: recall the trick that given a complex number $z \neq 1$, then $1 + z + z^2 + \dots + z^{n-1} = \frac{1-z^n}{1-z}$.*)
- (c) Compute the matrix $[USU^\dagger]$. What do you see?
- (d) The *discrete derivative operator* $\partial : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the linear map given by

$$(\partial v)_x = (v_x - v_{x-1}) \quad x = 0, 1, \dots, n-1.$$

Relate ∂ to the shift operator S and use this to show that $[U\partial U^\dagger]$ is also diagonal.

Problem 5. (*Spectrum and operators*)

Let $\mathcal{H}$ be a Hilbert space. Let us see what we can say about certain linear operators’ spectra (which refers to the set of “eigenvalues”).²

- (a) Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint, meaning $A = A^\dagger$. Show that every eigenvalue of A is real.
- (b) Let $P : \mathcal{H} \rightarrow \mathcal{H}$ be an orthogonal projection, meaning $P = P^\dagger = P^2$. Show that every eigenvalue of P must be either 0 or 1.
- (c) Let $U : \mathcal{H} \rightarrow \mathcal{H}$ be unitary, meaning $U^\dagger U = U U^\dagger = \mathbb{1}$. Show every eigenvalue of U must lie on the unit circle.

²Technically there are other types of spectra, but we often abuse the term eigenvalue in physics.

Problem 6. (*Examples from integral operators*)

A very useful class of linear operators comes from integral operators $I_K : L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$, which are defined by taking a function $K : [-\pi, \pi] \times [-\pi, \pi] \rightarrow \mathbb{C}$ and writing³

$$(I_K(f))(x) = \int_{-\pi}^{\pi} K(x, y) f(y) dy \quad \text{for all } f \in L^2([-\pi, \pi]).$$

- (a) Let $K(x, y) = |x - y|$. Show that I_K is self-adjoint.
- (b) Let $\psi \in L^2([-\pi, \pi])$ with $\|\psi\| = 1$ and define $K(x, y) = \psi(x)\overline{\psi(y)}$. Show that I_K is an orthogonal projection onto the linear span of ψ .

Problem 7. (*Translations and Derivatives*)

- (a) Fix any $\theta \in (-\pi, \pi)$. Show that the translation operator $S_\theta : L^2([-\pi, \pi]) \rightarrow L^2([-\pi, \pi])$ defined below, is unitary:

$$(S_\theta f)(x) = f(x - \theta), \quad x \in [-\pi, \pi],$$

where we interpret $x - \theta$ as $x - \theta \bmod 2\pi$ (for example, if $x = \pi/2$ and $\theta = -\pi$, then $x - \theta = 3\pi/2 = -\pi/2 \bmod 2\pi$).

- (b) Consider the derivative operator defined by

$$(\partial_\theta f)(x) = \lim_{\theta \rightarrow 0} \frac{f(x) - (S_\theta f)(x)}{\theta}$$

on smooth functions $f \in L^2([-\pi, \pi])$. Show that this operator is skew-adjoint, meaning for all smooth $f, g \in L^2([-\pi, \pi])$, we have

$$\langle f, \partial_\theta g \rangle = \langle (-\partial_\theta) f, g \rangle.$$

(For the purposes of this problem, you are allowed to freely pass limits through the inner product like so: $\langle \lim_{\theta \rightarrow 0} F(\theta), G \rangle = \lim_{\theta \rightarrow 0} \langle F(\theta), G \rangle$.)

³It may be useful to think of $K(x, y)$ as being like an “infinite matrix” whose entries are indexed by coordinates $(x, y) \in [-\pi, \pi] \times [-\pi, \pi]$.